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Quantum chaos of SU_3 observables

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Abstract. Hamiltonians built of SU_3 generators are presented which allow for qualitatively different classical limits. The generators of the group SU_3 give eight independent observables. In the classical limit two invariant functions of these observables, the so-called Casimir functions c_2 and c_3, take fixed values in the ranges $\frac{1}{2} \leqslant c_2 \leqslant \frac{2}{3}$ and $-\frac{2}{9} \leqslant c_3 \leqslant \frac{2}{9}$ (an SU_2 system, in contrast, has just one such Casimir function, the squared total angular momentum). Generically, this leads to a six-dimensional phase space. However, for two special ('degenerate') pairs of values $c_2 = \frac{2}{3}, c_3 = -\frac{2}{9}$ and $c_2 = \frac{2}{3}, c_3 = \frac{2}{9}$ the classical dynamics is only four-dimensional. One and the same Hamilton function of the generators may be integrable in the four-dimensional phase space but generate chaos in the six-dimensional case. We give two examples of Hamiltonians for which that alternative does arise; one of these is closely related to the Lipkin Hamiltonian familiar from nuclear shell models. The transition from integrable to chaotic classical behaviour is accompanied by the usual quantum transition from level clustering to level repulsion.

1. Introduction

There is a well established connection between the statistical properties of spectra of quantum systems with their classical counterparts. Generic classically nonintegrable systems are, on the quantum level, characterized by level repulsion, as well described by random-matrix theory. Classical integrability for two or more degrees of freedom, on the other hand, tends to go with level clustering [1–3].

In this paper we investigate a special class of systems which on the classical level have a compact phase space and, as a quantum mechanical consequence, a finite-dimensional Hilbert space. The classical limit is approached by increasing the dimension of the Hilbert space.

The simplest example of this kind is given by the dynamics of an angular momentum or spin. Due to the formal equivalence between a spin-$\frac{1}{2}$ and a two-level atom, collective dynamics of any number of identical two-level atoms can be described in terms of an angular momentum. Quantum signatures of chaos have been investigated in such systems for a long time (the kicked top [3]). When the number of atoms grows, the associated spin dynamics approaches a well defined classical limit. As is well known, the classical phase space of an angular momentum is two-dimensional (the surface of a sphere), corresponding to a single degree of freedom. To allow for chaos, periodic driving must be allowed for, as is indeed the case for the kicked top. Autonomous chaotic dynamics would require at least two coupled angular momenta [4, 5].

The basic symmetry for a three-level atom [6] is SU_3. In that case the algebra of observables is spanned by eight quantities, the generators of the Lie group SU_3. The Hilbert space of the quantum system is again finite-dimensional. In fact, if several identical three-level atoms are to be described, the Hilbert space dimension depends on the representation of SU_3 relevant for the actual configuration of the system. Irreducible representations of SU_3 are indexed by two independent quantum numbers (in contrast to the case of the angular momentum where a single quantum number, the total spin j, suffices to uniquely characterize the representation). There are thus many different ways of increasing the Hilbert space dimension towards a classical limit. We shall show that even the dimension of the classical phase space depends on the method of selection: the classical dynamics can involve either two or three degrees of freedom.

An interesting alternative could thus arise for autonomous dynamics with a Hamiltonian operator built by SU_3 generators. Such a given Hamiltonian could yield integrable classical dynamics in the four-dimensional phase space and chaotic motion in six dimensions. As the quantum analogue of that alternative one might expect, for one and the same quantum Hamiltonian operator, level clustering and level repulsion, depending on the representation chosen.

We shall, indeed, present two examples of Hamiltonians which do have very different spectral statistics in different irreducible representations. Level clustering is observed for irreducible representations that lie on the path to integrable classical dynamics and level repulsion sets in which the irreducible representation lies on the path to non-integrable dynamics.

Even though we have not selected our example Hamiltonians according to easy experimental realizability, it is clear that switching between qualitatively different dynamics in a given SU_3 system should be observable. At any rate, one of our example Hamiltonians is familiar from nuclear shell models [7–10]. Realizations in super-radiant lasers with many collectively pumped and radiating three-level atoms [11] are also imaginable†.

2. Quantum mechanics of a large number of three-level atoms with collective interaction

2.1. Quantum mechanical description of a single three-level atom

The Hilbert space of a single three-level atom is spanned by the orthonormal vectors

$$|1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad |2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad |3\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{2.1}$$

They are connected by the nine linear operators

$$\tau_{ij} = |i\rangle\langle j| \qquad i, j = 1, 2, 3. \tag{2.2}$$

The operator τ_{ij} describes a transition from the state $|j\rangle$ to the state $|i\rangle$, and its expectation value has the meaning of a complex polarization or coherence for $i \neq j$, and of an occupation probability in the diagonal case $i = j$. We incur the commutation relations‡

$$[\tau_{ij}, \tau_{kl}] = \delta_{kj}\tau_{il} - \delta_{il}\tau_{kj}. \tag{2.3}$$

The canonical group of basis transformations is SU_3, i.e. the group of unitary 3×3 matrices with unit determinant (an overall phase shift of basis vectors has been excluded here as it leaves all physical properties of basis vectors unchanged).

Any product of the operators τ_{ij} can be expanded in a linear combination using

$$\tau_{ij}\tau_{kl} = |i\rangle\langle j||k\rangle\langle l| = \delta_{kj}\tau_{il}. \tag{2.4}$$

† Though not closely related to our work, we would like to draw attention to a paper on inter- and intra-shell chaos in the hydrogen atom [12], where the relevant Hilbert spaces (the hydrogen shells) have finite dimension.

‡ These are the commutator relations of the Lie algebra $\mathfrak{gl}_3(\mathbb{C})$ (restricted to Hermitian generators it gives the Lie algebra $\mathfrak{u}_3$).

Whatever Hamiltonian H may be at work, the Heisenberg equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\tau_{ij} = \frac{\mathrm{i}}{\hbar}[H, \tau_{ij}] \tag{2.5}$$

is thus linear in the operators τ_{kl} and can be easily integrated.

2.2. Many three-level atoms with collective interaction

For N identical but distinguishable three-level atoms (to be referred to simply as three-level atoms hereafter) the atomic state is a vector in the N-fold tensor product of the Hilbert space of a single atom. We then incur collective coherences and occupation numbers of the atoms,

$$S_{ij} = \sum_{l=1}^{N} \tau_{ij}^{(l)}. \tag{2.6}$$

Here $\tau_{ij}^{(l)}$ acts as τ_{ij} in the Hilbert space of the lth atom (and as unity in the Hilbert spaces of all other atoms). The operators S_{ij} obey the same commutator relations (2.3) as the τ_{ij}†. The diagonal operators S_{ii} fulfil

$$S_{11} + S_{22} + S_{33} = N. \tag{2.7}$$

We may, therefore, restrict our attention to operators depending only on differences of the diagonal generators. In common usage in particle physics are the combinations

$$\begin{aligned} T_3 &:= \tfrac{1}{2}(S_{11} - S_{22}) \\ Y &:= \tfrac{1}{3}(S_{11} + S_{22} - 2S_{33}) \end{aligned} \tag{2.8}$$

usually referred to as the third components of isospin and hypercharge; we shall adhere to this notation although its physical significance, in our case, amounts to no more than a difference in occupation numbers. At any rate, the isospin and hypercharge variables together with the six coherences S_{ij}, with $i \neq j$, span the Lie algebra of the group SU_3.

We shall consider quantum mechanical systems built up by a fixed large number N of three-level atoms with a Hamiltonian depending only on the collective operators T_3, Y and $S_{ij} (i \neq j)$:

$$H = H(T_3, Y, S_{ij}) \qquad i \neq j \quad H = H^{\dagger}. \tag{2.9}$$

Interesting phenomena occur if we allow for Hamiltonians nonlinear in the generators, such as

$$H_{\mathrm{Lipkin}} = a(S_{11} - S_{33}) + b \sum_{i,j=1, i\neq j}^{3} S_{ij}^2 \tag{2.10}$$

which is known as the Lipkin Hamiltonian used in nuclear shell models [7–10]. Such Hamilton operators can also arise for three-level atoms collectively coupled to eigenmodes of the electromagnetic field in a cavity. They are block diagonal and each block is connected to an irreducible representation of the Lie group SU_3. From now on we shall always assume that the Hamilton matrix has been reduced to one of these irreducible blocks.

† The S_{ij} span a representation of the Lie algebra $\mathfrak{gl}_3(\mathbb{C})$.

2.3. Irreducible SU_3 representations

In this section we recall the most important facts on irreducible representations of the Lie group SU_3. Readers familiar with basic representation theory should skip this section.

We start with some words on SU_2 and its generators, angular momenta. The familiar irreducible representations are labelled by the one-integer or half-integer number j. As generators we may take three linear operators (matrices) J_+, J_-, J_3 in a $(2j + 1)$-dimensional Hilbert space with the well known angular momentum commutator relations

$$[J_+, J_-] = 2J_3 \qquad [J_3, J_\pm] = J_\pm. \tag{2.11}$$

A basis of the Hilbert space is provided by the eigenstates of J_3:

$$J_3|m; j\rangle = m|m; j\rangle \qquad m = -j, -j + 1, \dots, j - 1, j \tag{2.12}$$

and the number j is related to the squared total angular momentum operator $J^2 = J_3(J_3 + 1) + J_- J_+$, which is a constant in an irreducible representation $J^2 = j(j + 1)\mathbb{I}$. The number j also characterizes an irreducible representation as being the highest eigenvalue of J_3:

$$J_3|j\rangle_j = j|j; j\rangle. \tag{2.13}$$

The corresponding eigenvector $|j; j\rangle$ is usually called the highest-weight vector.

Just as for SU_2, any irreducible representation of SU_3 is uniquely characterized by its highest-weight vector $|\mu\rangle$. This highest-weight vector is unique up to a phase factor and defined as a common eigenstate of T_3 and Y with eigenvalues

$$\begin{aligned} T_3|\mu\rangle &= \frac{\lambda_1}{2}|\mu\rangle \\ Y|\mu\rangle &= \frac{\lambda_1 + 2\lambda_2}{3}|\mu\rangle \end{aligned} \tag{2.14}$$

and it is annihilated by raising operators,

$$S_{12}|\mu\rangle = S_{23}|\mu\rangle = S_{13}|\mu\rangle = 0. \tag{2.15}$$

The non-negative integers $\lambda_1, \lambda_2 = 0, 1, 2, \dots$ are used as labels for the irreducible SU_3 representations, just as the quantum number j is used to label irreducible SU_2 representations. We shall denote the representation characterized by λ_1 and λ_2 as $[\lambda_1; \lambda_2]$.

In any representation $[\lambda_1; \lambda_2]$ with either $\lambda_1 = 0$ or $\lambda_2 = 0$, the two operators T_3 and Y build a complete set of commuting observables, i.e. there is a basis of eigenvectors such that the eigenvalues of T_3 and Y are not degenerate. In a general representation $[\lambda_1; \lambda_2]$ with $\lambda_1 \neq 0$ and $\lambda_2 \neq 0$ such a basis is degenerate and the smallest complete set of commuting observables consists of the three operators T_3, Y and $T^2 = T_3(T_3 + 1) + S_{21}S_{12}$ (we shall call T^2 the total squared isospin).

The alternative just described for the number of operators in the smallest complete set of commuting observables will turn out to correspond to an alternative for the dimensionality of the phase space in the classical limit. Every operator in this commuting set gives one classical freedom. The classical limit thus yields a four-dimensional phase space for irreducible representations with one of the λ_i vanishing, while six dimensions arise if neither of the λ_i vanishes.

The construction of bases as well as matrices representing the operators S_{ij} was pioneered by Gelfand and Zetlin for SU_n and SO_n [13], see also [14, 15].

In the SU_2 case an irreducible representation is uniquely identified by the value of the squared angular momentum J^2, an operator commuting with all generators. A similar characterization of irreducible representations can be made for SU_3. To this end we need two independent operators commuting with all SU_3 generators, the so-called Casimir operators. In terms of the generators S_{ij}, these read

$$C_2 = \sum_{i,j=1}^{3} S_{ij} S_{ji} \tag{2.16}$$

$$C_3 = \sum_{i,j,k=1}^{3} S_{ij} S_{jk} S_{ki}. \tag{2.17}$$

In the Hilbert space of an irreducible representation $[\lambda_1; \lambda_2]$ they act as multiples of the identity operator

$$C_2 = a(\lambda_1, \lambda_2)\mathbb{I} \qquad C_3 = b(\lambda_1, \lambda_2)\mathbb{I}. \tag{2.18}$$

The constants a and b uniquely determine (and are uniquely determined by) the values of λ_1 and λ_2:

$$a(\lambda_1, \lambda_2) = \tfrac{2}{3}(\lambda_1^2 + \lambda_1\lambda_2 + \lambda_2^2 + 3\lambda_1 + 3\lambda_2) \tag{2.19}$$

$$b(\lambda_1, \lambda_2) = \tfrac{1}{9}(2\lambda_1^3 - 2\lambda_2^3 + 3\lambda_1^2\lambda_2 - 3\lambda_1\lambda_2^2 + 18\lambda_1^2 + 9\lambda_1\lambda_2 + 36\lambda_1 + 18\lambda_2) \tag{2.20}$$

and, therefore, also provide a unique characterization of each irreducible representation.

3. Classical limit

3.1. Construction of the classical limit via SU_3 coherent states

To perform the classical limit we must specify quantities s_{ij} corresponding to the quantum generators S_{ij}, calculate their Poisson brackets, characterize the phase space, and, finally, concretize the technically empty '$\hbar \to 0$' to a meaningful limit for controllable parameters. In particular, a quantum dynamics generated by a Hamiltonian of the form

$$H = \sum_{l_1,l_2=1}^{3} a^{(1)}_{l_1 l_2} \hbar S_{l_1 l_2} + \sum_{l_1,l_2,l_3,l_4=1}^{3} a^{(2)}_{l_1 l_2 l_3 l_4} \hbar^2 S_{l_1 l_2} S_{l_3 l_4} + \cdots \tag{3.1}$$

with coefficients $a^{(i)}$ independent of $\hbar$ will then yield a classical dynamics generated by an associated Hamiltion function H^{cl} with the s_{ij} as independent variables.

In the case of SU_2, the dimension $2j + 1$ of irreducible representations can be taken as a controllable representative of $1/\hbar$. The classical limit arises by sending the dimension to infinity. It is natural to proceed analogously for SU_3. However, since the dimension of an irreducible representation here depends on the two integer parameters λ_1 and λ_2 as $d = \frac{(\lambda_1+1)(\lambda_2+1)(\lambda_1+\lambda_2+2)}{2}$, there are many possibilities of approaching the classical limit via different irreducible representations with increasing dimensions. We shall eventually be led to choosing the representative of $\hbar$ as $\hbar = (\lambda_1 + \lambda_2)^{-1}$.

In an earlier paper [16] two of us constructed the classical limit and the classical phase space using SU_3 coherent states. Referring the reader to [16] for details here we repeat only what is needed for our further considerations.

Non-normalized coherent states are found for each non-degenerate ($\lambda_1; \lambda_2 \neq 0$) irreducible representation $[\lambda_1; \lambda_2]$ by acting on the highest weight vector $|\mu\rangle$ (2.15), (2.14) with all elements of the group of lower-triangular complex matrices as [17, 18]

$$||\gamma\rangle = b(\gamma)|\mu\rangle \tag{3.2}$$

$$b(\gamma) = \exp(\gamma_3 S_{31}) \exp(\gamma_1 S_{21}) \exp(\gamma_2 S_{32}) \tag{3.3}$$

with the shorthand notation $\gamma = (\gamma_1, \gamma_2, \gamma_3) \in \mathbb{C}^3$.

To normalize the states we calculate the norm

$$\langle\gamma||\gamma\rangle = f_1^{\lambda_1} f_2^{\lambda_2} \tag{3.4}$$

obtain

$$f_1 = 1 + |\gamma_1|^2 + |\gamma_3|^2 \tag{3.5}$$

$$f_2 = 1 + |\gamma_2|^2 + |\gamma_3 - \gamma_1\gamma_2|^2 \tag{3.6}$$

and define the normalized coherent states by

$$|\gamma\rangle = f_1^{-\lambda_1/2} f_2^{-\lambda_2/2} ||\gamma\rangle. \tag{3.7}$$

The coherent states make up a SU_3-invariant submanifold of Hilbert space, i.e.

$$g|\gamma\rangle = e^{i\alpha(\gamma,g)}|\gamma'(\gamma, g)\rangle \qquad \text{for} \quad g = \exp\left(i\sum a_{kl}S_{kl}\right) \in SU_3. \tag{3.8}$$

The coherent states build a manifold with real dimension six (for $\lambda_1, \lambda_2 \neq 0$) or four (for $\lambda_1 = 0$ or $\lambda_2 = 0$) [16]. In the classical limit the submanifold will become, by appropriate rescaling, the classical phase space. Turning to the six-dimensional case first, we now proceed to defining observables and Poisson brackets on the manifold of coherent states.

As classical observables s_{ij} we take the expectation values of the generators S_{ij} with respect to coherent states. Straightforward calculations give

$$\begin{aligned}
s_{32} &= \hbar\langle\gamma|S_{32}|\gamma\rangle = \hbar\frac{\lambda_1}{f_1}\gamma_1\gamma_3^* + \hbar\frac{\lambda_2}{f_2}\gamma_2^* \\
s_{31} &= \hbar\langle\gamma|S_{31}|\gamma\rangle = \hbar\frac{\lambda_1}{f_1}\gamma_3^* + \hbar\frac{\lambda_2}{f_2}(\gamma_3^* - \gamma_1^*\gamma_2^*) \\
s_{21} &= \hbar\langle\gamma|S_{21}|\gamma\rangle = \hbar\frac{\lambda_1}{f_1}\gamma_1^* - \hbar\frac{\lambda_2}{f_2}\gamma_2(\gamma_3^* - \gamma_1^*\gamma_2^*) \\
s_{23} &= \hbar\langle\gamma|S_{23}|\gamma\rangle = \hbar\frac{\lambda_1}{f_1}\gamma_1^*\gamma_3 + \hbar\frac{\lambda_2}{f_2}\gamma_2 \\
s_{13} &= \hbar\langle\gamma|S_{13}|\gamma\rangle = \hbar\frac{\lambda_1}{f_1}\gamma_3 + \hbar\frac{\lambda_2}{f_2}(\gamma_3 - \gamma_1\gamma_2) \\
s_{12} &= \hbar\langle\gamma|S_{12}|\gamma\rangle = \hbar\frac{\lambda_1}{f_1}\gamma_1 - \hbar\frac{\lambda_2}{f_2}\gamma_2^*(\gamma_3 - \gamma_1\gamma_2) \\
t_3 &= \frac{1}{2}(s_{11} - s_{22}) = \hbar\langle\gamma|T_3|\gamma\rangle = \hbar\frac{\lambda_1}{2f_1}(1 - |\gamma_1|^2) + \hbar\frac{\lambda_2}{2f_2}(|\gamma_2|^2 - |\gamma_3 - \gamma_1\gamma_2|^2) \\
y &= \frac{1}{3}(s_{11} + s_{22} - 2s_{33}) = \hbar\langle\gamma|Y|\gamma\rangle = \hbar\frac{\lambda_1}{3f_1}(1 + |\gamma_1|^2 - 2|\gamma_3|^2) \\
&\quad + \hbar\frac{\lambda_2}{3f_2}(2 - |\gamma_2|^2 - |\gamma_3 - \gamma_1\gamma_2|^2).
\end{aligned} \tag{3.9}$$

The functions s_{ij} should have Poisson brackets compatible with the commutators of their quantum correspondents S_{ij},

$$\begin{aligned}
\{s_{ij}, s_{kl}\} &:= i\hbar\langle\gamma|[S_{ij}, S_{kl}]|\gamma\rangle \\
&= i\delta_{kj}s_{il} - i\delta_{il}s_{kj}.
\end{aligned} \tag{3.10}$$

This definition uniquely yields Poisson brackets for functions on the manifold of coherent states [16, 19]: any pair of complex valued functions $f(\gamma)$ and $g(\gamma)$ of the three complex variables $\gamma = (\gamma_1, \gamma_2, \gamma_3)$ obeys

$$\{f, g\} = \frac{i}{\hbar}\sum_{k,l=1}^{3}\omega^{kl}\left(\frac{\partial f}{\partial\gamma_l}\frac{\partial g}{\partial\gamma_k^*} - \frac{\partial f}{\partial\gamma_k^*}\frac{\partial g}{\partial\gamma_l}\right) \tag{3.11}$$

where the coefficients ω^{kl} are given by†

$$
\begin{aligned}
\omega^{11} &= \frac{f_1}{(\lambda_1+\lambda_2)}\left((1+|\gamma_1|^2)+\frac{\lambda_2}{\lambda_1}\frac{f_1}{f_2}\right)\\
\omega^{12} &= \frac{1}{(\lambda_1+\lambda_2)}(\gamma_1^*+\gamma_2\gamma_3^*)(\gamma_1^*\gamma_3-\gamma_2-\gamma_2|\gamma_1|^2)\\
\omega^{13} &= \frac{f_1}{(\lambda_1+\lambda_2)}\left(\gamma_1^*\gamma_3+\frac{\lambda_2}{\lambda_1}\frac{f_1}{f_2}\gamma_2\right)\\
\omega^{21} &= \frac{1}{(\lambda_1+\lambda_2)}(\gamma_1+\gamma_2^*\gamma_3)(\gamma_1\gamma_3^*-\gamma_2^*-\gamma_2^*|\gamma_1|^2)\\
\omega^{22} &= \frac{f_2}{(\lambda_1+\lambda_2)}\left((1+|\gamma_2|^2)+\frac{\lambda_1}{\lambda_2}\frac{f_2}{f_1}\right)\\
\omega^{23} &= \frac{1}{(\lambda_1+\lambda_2)}(\gamma_1+\gamma_2^*\gamma_3)(1+|\gamma_3|^2-\gamma_1^*\gamma_2^*\gamma_3)\\
\omega^{31} &= \frac{f_1}{(\lambda_1+\lambda_2)}\left(\gamma_1\gamma_3^*+\frac{\lambda_2}{\lambda_1}\frac{f_1}{f_2}\gamma_2^*\right)\\
\omega^{32} &= \frac{1}{(\lambda_1+\lambda_2)}(\gamma_1^*+\gamma_2\gamma_3^*)(1+|\gamma_3|^2-\gamma_1\gamma_2\gamma_3^*)\\
\omega^{33} &= \frac{f_1}{(\lambda_1+\lambda_2)}\left((1+|\gamma_3|^2)+\frac{\lambda_2}{\lambda_1}\frac{f_1}{f_2}|\gamma_2|^2\right).
\end{aligned}
\tag{3.12}
$$

A completely analogous construction can also be given in the degenerate case when $\lambda_1 = 0$ or $\lambda_2 = 0$. We should only restrict the group element $b(\gamma)$ used to generate the coherent states in (3.3) to $\gamma_2 = 0$ for $\lambda_1 = 0$ and to $\gamma_1 = 0$ for $\lambda_2 = 0$. In these cases we end up with four-dimensional manifolds parametrized, respectively, by (γ_1, γ_3) or (γ_2, γ_3). All of the above results on expectation values and Poisson bracket remain valid, once the mentioned restrictions on γ_i and λ_i are inserted.

As already mentioned, the classical limit amounts to taking the dimension of the irreducible representation to infinity [23]; but this can be done in many ways due to the fact that the dimension depends on the two parameters λ_1 and λ_2. We will keep the ratio

$$q = \frac{\lambda_1}{\lambda_1+\lambda_2} = \hbar\lambda_1 \tag{3.13}$$

fixed while both λ_1 and λ_2 go to infinity and thus $\hbar = (\lambda_1+\lambda_2)^{-1}$ to zero. Let us thus take a series of irreducible representations $\{[n\lambda_1^{(0)}, n\lambda_2^{(0)}]\}_{n=1,2,3,\ldots}$ with fixed integers $\lambda_1^{(0)}, \lambda_2^{(0)}$ and

$$\hbar = \frac{1}{(\lambda_1^{(0)}+\lambda_2^{(0)})n} \xrightarrow{n\to\infty} 0. \tag{3.14}$$

Since in any irreducible representation the ratio q is fixed, there is a unique correspondence to the sequence of irreducible representations that define the classical limit. It is easy to see that the Poisson brackets have a well defined limit under (3.14), as do our classical observables $s_{ij} = \langle\gamma|\hbar S_{ij}|\gamma\rangle$. Moreover, expectation values of products of operators factorize as follows:

$$\langle\gamma|\mathcal{O}_1\mathcal{O}_2|\gamma\rangle \longrightarrow \langle\gamma|\mathcal{O}_1|\gamma\rangle\langle\gamma|\mathcal{O}_2|\gamma\rangle \qquad \text{for} \quad \hbar\to 0 \tag{3.15}$$

and the connection between the Poisson bracket and the commutator (3.10) is no longer limited to the observables linear in the generators,

$$\frac{\mathrm{i}}{\hbar}\langle\gamma|[\hbar S_{ij}\hbar S_{kl}, \hbar S_{mn}]|\gamma\rangle \xrightarrow{\hbar\to 0} s_{ij}\{s_{kl}, s_{mn}\} + \{s_{ij}, s_{mn}\}s_{kl} = \{s_{ij}s_{kl}, s_{mn}\}. \tag{3.16}$$

† It is not trivial to calculate the coefficients ω^{kl} directly from definition (3.10). Such a calculation should be performed using a SU_3-invariant symplectic form on the manifold of coherent states. There is a well founded mathematical theory of symplectic structures (and Poisson brackets) on manifolds of coherent states for various classes of Lie groups [19, 21, 22]. A thorough mathematical introduction is found in [19], while in [16] the construction for SU_3 is presented in detail.

As a consequence, the classical observables evolve in the six-dimensional phase space according to the Hamilton equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{o}\,(t) = \{H^{cl}, \mathrm{o}\,(t)\} \tag{3.17}$$

with the Hamilton function

$$\begin{aligned} H^{cl} &= \lim_{\hbar\to 0}\langle\gamma|H|\gamma\rangle \\ &= \sum_{l_1,l_2=1}^{3} a^{(1)}_{l_1 l_2} s_{l_1 l_2} + \sum_{l_1,l_2,l_3,l_4=1}^{3} a^{(2)}_{l_1 l_2 l_3 l_4} s_{l_1 l_2} s_{l_3 l_4} + \cdots \end{aligned} \tag{3.18}$$

calculated as the limit of the expectation value of the quantum Hamiltonian (3.1).

It is obvious that going to the limit via a series of degenerate representations (e.g. keeping $\lambda_1 = 0$ in (3.14)) we formally obtain the same results concerning the form of classical equations of motion but in the four-dimensional phase space.

The different ways of approaching the classical limit can be interpreted for our model of many three-level atoms. Obviously, for an arbitrary large number of atoms it is always (at least in principle) possible to prepare an initial state belonging to some specific representation of the SU_3 symmetry group of the Hamiltonian; the subsequent evolution will then be restricted to that subspace. For instance, any fully symmetric initial state will evolve solely in the $((N+1)(N+2)/2)$-dimensional space of the degenerate representation with $\lambda_1 = N$ and $\lambda_2 = 0$. Other initial states will lead to other parameters λ_1, λ_2. Hence approaching the classical limit corresponds to increasing the number of atoms and preparing them in the appropriate initial states. Analogous considerations for a model of a superradiant laser involving many three-level atoms were presented in [11].

3.2. The dimensionality of the classical phase space

Here we shall explain the occurrence of different dimensionalities for the classical phase space of a SU_3 system in purely classical terms. The connection with the classical limit described above will be drawn at the end of this section.

Let us consider the classical observables $s_{ij} (i, j = 1, 2, 3)$ restricted by

$$\begin{aligned} s_{ij} &= s^*_{ji} \\ s_{11} + s_{22} + s_{33} &= 0 \end{aligned} \tag{3.19}$$

and assume their Poisson brackets to be those of the generators of the Lie group SU_3,

$$\{s_{kl}, s_{mn}\} = \mathrm{i}(\delta_{ml} s_{kn} - \delta_{kn} s_{ml}). \tag{3.20}$$

Moreover, we assume the dynamics of these observables generated by some Hamilton function $H^{cl} = H^{cl}(s_{ij})$ and Hamilton's equations

$$\frac{\mathrm{d}}{\mathrm{d}t} s_{ij} = \{H^{cl}, s_{ij}\}. \tag{3.21}$$

It is now convenient to consider the observables s_{ij} to be matrix elements of a traceless Hermitian matrix $\mathcal{S}$. The traceless Hermitian 3×3 matrices span an eight-dimensional manifold in the space $\mathbb{M}_{3\times3}(\mathbb{C})$ of all complex 3×3 matrices†. Using the Poisson brackets (3.20) it is

† In general, if G is a Lie group, and $\mathfrak{g}$ its Lie algebra, there is a canonical Poisson bracket structure on $\mathfrak{g}^*$, the dual vector space of $\mathfrak{g}$ [19]. In our case we can identify $\mathfrak{su}_3{}^*$ with $\mathfrak{su}_3$, both represented as the space of Hermitian traceless 3×3 matrices and the canonical Poisson structure is simply the one presented above.

easy to check that the two Casimir functions

$$\begin{aligned} c_2 &= \operatorname{tr}\mathcal{S}^2 = \sum_{i,j=1}^{3} s_{ij}s_{ji} \\ c_3 &= \operatorname{tr}\mathcal{S}^3 = \sum_{i,j,k=1}^{3} s_{ij}s_{jk}s_{ki} \end{aligned} \tag{3.22}$$

are constants of motion for any Hamilton function $h(s_{ij})$. Since any dynamics is restricted to the hypersurfaces $c_2 = \text{const}$ and $c_3 = \text{const}$, we can consider one intersection of these hypersurfaces. Each intersection for given constants $c_2^{(0)}, c_3^{(0)}$ defines a phase space $\mathcal{M}$ for a SU_3 system†

$$\mathcal{M} = \{\mathcal{S} \in \mathbb{M}_{3\times3}(\mathbb{C}) : \mathcal{S} = \mathcal{S}^\dagger, \operatorname{tr}\mathcal{S} = 0, \operatorname{tr}\mathcal{S}^2 = c_2^{(0)}, \operatorname{tr}\mathcal{S}^3 = c_3^{(0)}\}. \tag{3.23}$$

Generically, such an intersection will be six-dimensional. However, a closer look reveals that for some values of the Casimir functions the phase space may have smaller dimension. To see this let us observe that we can use the eigenvalues ν_1, ν_2, ν_3 of the matrix $\mathcal{S}$ instead of the Casimir invariants to define the classical phase space (3.23). Indeed, the characteristic equation for a traceless Hermitian matrix reads [24]

$$\det(\mathcal{S} - \nu) = -\nu^3 - \tfrac{1}{2}\operatorname{tr}\mathcal{S}^2\nu + \tfrac{1}{3}\operatorname{tr}\mathcal{S}^3 = (\nu_1 - \nu)(\nu_2 - \nu)(\nu_3 - \nu). \tag{3.24}$$

We see that the coefficients of the characteristic equation are just the Casimir functions. So there is a one-to-one correspondence between the eigenvalues ν_1, ν_2, ν_3 of $\mathcal{S}$ and the values of the Casimir functions. As a consequence, we can rewrite the manifold (3.23) as

$$\mathcal{M} = \{\mathcal{S} \in \mathbb{M}_{3\times3}(\mathbb{C}) : \mathcal{S} = \mathcal{S}^\dagger,\ \text{eigenvalues } (\nu_1, \nu_2, \nu_3), \nu_1 + \nu_2 + \nu_3 = 0\} \tag{3.25}$$

such that the phase space may be said to consist of all complex, Hermitian, traceless matrices, all having the same eigenvalues ν_1, ν_2, ν_3, with $\nu_1 + \nu_2 + \nu_3 = 0$.

With this characterization at hand it is now easy to count the dimensions of the phase spaces. Any matrix $\mathcal{S}$ with given eigenvalues ν_i may be written as a conjugation

$$\mathcal{S} = U \begin{pmatrix} \nu_1 & 0 & 0 \\ 0 & \nu_2 & 0 \\ 0 & 0 & \nu_3 \end{pmatrix} U^\dagger = U D(\nu_i) U^\dagger \tag{3.26}$$

where U is some matrix from the group U_3. We see, however, that in (3.26) the correspondence between $\mathcal{S}$ and U is not one to one. Indeed, if U fulfils (3.26) then the same is true for $U' = UU_1$ if only U_1 leaves the diagonal matrix $D(\nu_i)$ invariant, i.e. $U_1 D(\nu_i) U_1^\dagger = D(\nu_i)$ (or, in other words, U_1 and $D(\nu_i)$ commute). To have a unique characterization of $\mathcal{S}$ and U we have to divide out the matrices commuting with $D(\nu_i)$ from the group U_3.

Let us first assume that the eigenvalues are nondegenerate, $\nu_1 \neq \nu_2 \neq \nu_3$. Then the only matrices $U \in U_3$ commuting with the diagonal matrix $D(\nu_i)$ are diagonal matrices themselves. So in this case the phase space looks locally (near a diagonal matrix) like $U_3/(U_1 \times U_1 \times U_1) = SU_3/(U_1 \times U_1)$ (for the explicit form of the equivalence see [16]) which is a six-dimensional manifold.

If, however, any two of the eigenvalues are degenerate, say $\nu_1 = \nu_2$, then any matrix U which is of the form

$$U = \begin{pmatrix} \tilde{U} & 0 \\ 0 & 1 \end{pmatrix} \tag{3.27}$$

† These phase spaces are symplectic manifolds known as coadjoint orbits of the Lie group SU_3, see [19].

where $\tilde{U}$ is a 2×2 matrix from U_2 will leave the diagonal matrix $D(\nu_i)$ invariant. This shows that phase space is locally like $U_3/(U_2 \times U_1) = SU_3/(SU_2 \times U_1)$, which is a four-dimensional manifold (again global equivalence may be proven).

There are no other nontrivial possibilities. The phase space is thus either six-dimensional if the eigenvalues ν_i are not degenerate or four-dimensional for degenerate eigenvalues.

The foregoing classical consideration should now be put into the perspective of quantum mechanics and the classical limit. To that end we remark that the same group quotients $SU_3/(U_1 \times U_1)$ and $SU_3/(SU_2 \times U_1)$ appear in the definition of SU_3 coherent states. This is, of course, no accident. In a nondegenerate irreducible representation (i.e. $\lambda_1 \neq 0$ and $\lambda_2 \neq 0$) the highest-weight vector is invariant under the action of the subgroup $U_1 \times U_1$ of SU_3 (a change in the phase of the vector is admitted for invariance here), whereas for the degenerate representations (either $\lambda_1 = 0$ or $\lambda_2 = 0$) the subgroup $SU_2 \times U_1$ comes in. In fact, the SU_3 coherent states are a symplectic submanifold of the corresponding Hilbert space and, as such, isomorphic to the phase space attained in the classical limit (mathematicians have developed the theory of geometric quantization which focuses on this isomorphism [18, 20–22]).

In section 3 we defined the classical observables s_{ij} and their Poisson brackets in an appropriate limit $\hbar \longrightarrow 0$. The structure is obviously the same as the one considered in the present section. The limit was performed while keeping fixed a parameter which defines a sequence of irreducible representations. That parameter also determines the eigenvalues,

$$\nu_1 = \frac{q+1}{3} \qquad \nu_2 = \frac{1-2q}{3} \qquad \nu_3 = \frac{q-2}{3}. \tag{3.28}$$

This may be checked as follows: first, we realize that the eigenvalues ν_i of the matrix s_{ij} are independent of γ and may thus be calculated for $\gamma = 0$. Indeed, the coherent state $|\gamma\rangle$ may be represented as a 'rotated' version of the highest-weight state, $|\gamma\rangle = \mathrm{e}^{\mathrm{i}M(\gamma)}|\gamma = 0\rangle$, where $M(\gamma)$ is a Hermitian operator linear in the quantum generators S_{ij}; on momentarily introducing a real parameter τ to generalize the rotation to $\mathrm{e}^{\mathrm{i}\tau M(\gamma)}$ and differentiating w.r.t. τ we find that $s_{ij}(\gamma) = \sum_{kl} u_{ik}(\gamma)s_{kl}(0)u^{\dagger}_{lj}(\gamma)$, with a unitary 3×3 matrix u in which we may set $\tau = 1$; the unitarity of u entails $s_{ij}(\gamma)$ and $s_{ij}(0)$ to have the same eigenvalues. Next, for $\gamma = 0$ we have $s_{ij}(0) = \nu_i\delta_{ij}$, and thus from (3.9), $t_3 = (\nu_1 - \nu_2)/2$, $y = (\nu_1 + \nu_2 - 2\nu_3)/3$, $\nu_1 + \nu_2 + \nu_3 = 0$, while (2.14) and our definition (3.13) of the representative of $\hbar$ yield $t_3 = q/2$, $y = (2-q)/3$; these linear relations imply that q determines the eigenvalues ν_i as given in (3.28).

The classical Casimir functions are also assigned unique values,

$$\begin{aligned} c_2 &= \sum_{i,j=1}^{3} s_{ij}s_{ji} = \tfrac{2}{3}(q^2 - q + 1) \\ c_3 &= \sum_{i,j,k=1}^{3} s_{ij}s_{jk}s_{ki} = \tfrac{1}{9}(-2q^3 + 3q^2 + 3q - 2) \end{aligned} \tag{3.29}$$

while their ranges are, due to $0 \leqslant q \leqslant 1$,

$$\tfrac{1}{2} \leqslant c_2 \leqslant \tfrac{2}{3} \qquad -\tfrac{2}{9} \leqslant c_3 \leqslant \tfrac{2}{9}. \tag{3.30}$$

For $q = 0$ or $q = 1$ only degenerate irreducible representations are involved in the classical limit. The Casimir functions then take values at the boundaries of the admissible ranges according to

$$\left.\begin{array}{l} \nu_1 = \nu_2 = \frac{1}{3} \\ \nu_3 = -\frac{2}{3} \\ c_2 = \frac{2}{3} \\ c_3 = -\frac{2}{9} \end{array}\right\} \text{for} \quad q = 0 \qquad \text{and} \qquad \left.\begin{array}{l} \nu_1 = \frac{2}{3} \\ \nu_2 = \nu_3 = -\frac{1}{3} \\ c_2 = \frac{2}{3} \\ c_3 = \frac{2}{9} \end{array}\right\} \text{for} \quad q = 1. \tag{3.31}$$

So for $q = 0$ or $q = 1$ the classical limit leads to a phase space of dimension four. For all other values $0 < q < 1$ the classical phase space is six-dimensional.

3.3. Action-angle coordinates on the classical phase space

In the classical limit we had used the complex coordinates γ which nicely fit the group structure of coherent states. The Poisson brackets of these variables, however, do not have the canonical form

$$\{p_k, q_l\} = \delta_{kl} \qquad \text{with} \quad k, l = 1, 2, 3. \tag{3.32}$$

On the other hand, canonical pairs fulfilling (3.32) are desirable in the investigation of dynamical systems. For instance, if the Hamilton function does not depend on one canonical coordinate, say q_1, we immediately know that its conjugate p_1 is a constant of motion. We may then investigate a manifold of lower dimension spanned by the remaining pairs and regard p_1 as a parameter.

Canonical pairs may be constructed by considering the smallest complete set of commuting quantum observables (i.e. Y, T_3, T^2) in the classical limit. These give three Poisson commuting classical observables. The conjugate variables may then be found in different ways [16,25,26]. Foregoing the derivation we just give the transformation from the s_{ij} to canonical variables which we present as action-angle variables $p_k = I_k, q_k = \phi_k$, with $\phi \in [0, 2\pi)$:

$$\begin{aligned}
&y = I_1 \qquad t_3 = I_2 \qquad \sqrt{t_3^2 + s_{12}s_{21}} = I_3 \\
&s_{12} = \exp(\mathrm{i}\phi_2)\sqrt{I_3^2 - I_2^2} \\
&s_{13} = \exp\left[\frac{\mathrm{i}}{2}(2\phi_1 + \phi_2 + \phi_3)\right]\sqrt{I_3 + I_2}A_q + \exp\left[\frac{\mathrm{i}}{2}(2\phi_1 + \phi_2 - \phi_3)\right]\sqrt{I_3 - I_2}B_q \\
&s_{23} = \exp\left[\frac{\mathrm{i}}{2}(2\phi_1 - \phi_2 + \phi_3)\right]\sqrt{I_3 - I_2}A_q - \exp\left[\frac{\mathrm{i}}{2}(2\phi_1 - \phi_2 - \phi_3)\right]\sqrt{I_3 + I_2}B_q.
\end{aligned} \tag{3.33}$$

The coefficients A_q, B_q are defined by

$$\begin{aligned}
A_q &= \frac{1}{2I_3}\sqrt{\left(\frac{2q-1}{3} + I_3 + \frac{I_1}{2}\right)\left(\frac{2-q}{3} + I_3 + \frac{I_1}{2}\right)\left(\frac{q+1}{3} - I_3 - \frac{I_1}{2}\right)} \\
B_q &= \frac{1}{2I_3}\sqrt{\left(-\frac{2q-1}{3} + I_3 - \frac{I_1}{2}\right)\left(\frac{2-q}{3} - I_3 + \frac{I_1}{2}\right)\left(\frac{q+1}{3} + I_3 - \frac{I_1}{2}\right)}
\end{aligned} \tag{3.34}$$

and $q = \frac{\lambda_1}{\lambda_1+\lambda_2}$ is the parameter defining the sequence of irreducible representations used.

The consistency of the coordinate transformation (3.33) with the Poisson brackets for the functions s_{ij} may be checked by direct calculation.

The range of the action variables I_j in phase space is

$$\begin{aligned}
&I_1 \equiv y \in \left[-\frac{1+q}{3}, \frac{2-q}{3}\right] \\
&I_3 \equiv t \in \left[\max\left\{\frac{y}{2} - \frac{1-2q}{3}, \frac{1-2q}{3} - \frac{y}{2}\right\}, \min\left\{\frac{y}{2} - \frac{q-2}{3}, \frac{q+1}{3} - \frac{y}{2}\right\}\right] \\
&I_2 \equiv t_3 \in [-t, t].
\end{aligned} \tag{3.35}$$

4. Chaos and integrability in quantum and classical mechanics

In this section we present numerical results for two example systems which illustrate the dependence of level statistics on the irreducible representations (section 4.2). We also investigate the corresponding classical dynamics. We shall show that indeed one and the same quantum mechanical Hamilton operator may have different classical limits with qualitatively different dynamics. The different classical limits arise because we have freedom in choosing a sequence of irreducible representations for the classical limit (these sequences are labelled by the number $q = \frac{\lambda_1}{\lambda_1+\lambda_2} \in [0, 1]$, which is kept fixed in the limit). We had already shown in section 3 that different choices of q may even lead to different dimensions (four or six) of the classical phase space. Qualitative differences in the classical dynamics become obvious when a transformation to appropriate (i.e. canonical) variables is made (in the variables s_{ij} the Hamilton function looks the same for any choice of irreducible representations in the classical limit, but there are hidden constraints on these variables). The two systems investigated here both have one constant of motion independent of the Hamiltonian itself. So we know from the start that the four-dimensional classical limit is integrable. However, both examples will turn out to display chaos in the six-dimensional limit.

In section 3 we showed that to each irreducible representation there is a unique corresponding classical limit. This correspondence shows up in the level spacing distribution $P(s)$ of the quantum mechanical systems. In irreducible representations corresponding to a chaotic classical limit, $P(s)$ is a Wigner distribution, while classical integrability tends to go with Poissonian level statistics, i.e. an exponential $P(s)$.

4.1. Two examples

Our example Hamiltonians are

$$H_1 = \overline{h}T_3 + 3\overline{h}^2(S_{12}^2 + S_{21}^2) + 15\overline{h}^2(S_{13}S_{32} + S_{23}S_{31}) \tag{4.1}$$

$$H_2 = \sin(100\overline{h}T_3) + \tfrac{1}{2}\overline{h}^2(S_{12}^2 + S_{21}^2) + 100\overline{h}^2(S_{13}S_{32} + S_{23}S_{31}). \tag{4.2}$$

They both commute with the hypercharge Y

$$[H_i, Y] = 0 \qquad \text{for} \quad i = 1, 2 \tag{4.3}$$

so hypercharge is conserved. All quantum mechanical investigations have been performed in an eigenspace of Y.

The Hamiltonian H_1 resembles the Lipkin Hamiltonian (2.10). It is the simplest Hamiltonian we have found which shows strongly different level spacing distributions in different irreducible representations. Being only of second-order in the generators such a Hamiltonian could, in principle, be realized with three-level atoms in a resonator coupled to detuned modes.

The second Hamiltonian H_2 could hardly describe a real physical system because of the sine function contained. We have chosen to present that Hamiltonian here since we have found it to possess a beautiful transition from a Poissonian distribution of level spacings $P(s)$ (in the irreducible representations corresponding to an integrable classical limit) to the Wigner $P(s)$ (in other irreducible representations with a nonintegrable limit). Actually, the Poissonian statistics in the integrable case could not be expected since the restriction to an eigenspace of the hypercharge leaves us with but a single degree of freedom in the integrable case. Poissonian behaviour is really due to the sine function in the Hamiltonian: the function $\sin(kn)$ is somewhat similar to a (bad) random-number generator, provided the parameter k is not a rational multiple of π and $k \gg \pi$, as n runs through the integers.

In the classical limit we have set $\overline{h} = \frac{1}{\lambda_1+\lambda_2}$ and have kept $q = \overline{h}\lambda_1$ fixed as described in section 3. We then have $0 \leqslant q \leqslant 1$ and the classical systems are described by the Hamilton functions

$$H_1^{cl} = t_3 + 3(s_{12}^2 + s_{21}^2) + 15(s_{13}s_{32} + s_{23}s_{31}) \tag{4.4}$$

$$H_2^{cl} = \sin(100t_3) + \tfrac{1}{2}(s_{12}^2 + s_{21}^2) + 100(s_{13}s_{32} + s_{23}s_{31}) \tag{4.5}$$

which if written in terms of the variables $t_3, y, s_{ij} (i \neq j)$ seem independent of the irreducible representations used in the classical limit (i.e. independent of q). However, the two Casimir functions, $c_2 = \lim_{\hbar \to 0} \hbar^2 C_2$ and $c_3 = \lim_{\hbar \to 0} \hbar^3 C_3$, are conserved so that these eight numbers are not independent. Moreover, like its quantum counterpart Y, the classical hypercharge y is a constant of motion. The conservation of y guarantees the integrability of the two Hamiltonians above in the four-dimensional phase spaces.

Note that there is no free parameter other than q in the Hamiltonians. Thus our investigations rely on looking at one and the same Hamiltonian for different irreducible SU_3 representations.

4.2. Quantum chaos in different irreducible representations

For the numerical investigation of the level spacing statistics we have chosen an eigenspace of the hypercharge with eigenvalue $Y = \frac{\lambda_1 - \lambda_2}{3}$ of the Hilbert space of the $[\lambda_1; \lambda_2]$ representation. The dimension of this subspace is

$$d = (\lambda_1 + 1)(\lambda_2 + 1). \tag{4.6}$$

This eigenspace has been chosen because it has the largest dimension. For the first Hamiltonian H_1 only energy eigenvalues in the range $-2.0 \leqslant E \leqslant 0.5$ have been taken into account for the level statistics. In that range quantum signatures of chaos seem to show up best. For the second Hamiltonian we did not have to restrict ourselves to any energy range. The spectra have been unfolded such that the mean level spacing equals unity.

Figures 1 and 2 show the level spacing distribution $P(s)$ and, in the insets, its integral

$$N(s) = \int_0^s \mathrm{d}s' P(s'). \tag{4.7}$$

The thin curves show $P(s)$ and $N(s)$ of random matrix theory (Gaussian orthogonal ensemble (GOE)) and for a uncorrelated spectrum (Poisson). The Wigner distribution of the GOE is conjectured to be universal for classically chaotic systems. Uncorrelated spectra with $P(s)$ a Poisson distribution are known to be generic for classically integrable systems with more than one degree of freedom.

The first graph of figure 1 shows $P(s)$ and $N(s)$ for the Hamiltonian H_1 in an irreducible representation with $q = 1$ corresponding to a four-dimensional classical limit. Since this is basically a one-freedom system, Poisson behaviour is not expected and indeed $N(s)$ has three sharp steps at $s = 0, 1, 2$ indicating a very rigid regular spectrum. In the following graphs q is decreased. For $q = 0.99$, 0.98, 0.95 the sharp steps in $N(s)$ wash out and the distributions look much similar to a Poisson distribution with a surviving strong peak for $S = 0$ in $P(s)$. Since $0 < q < 1$ in these cases we have a six-dimensional classical limit so we cannot know *a priori* whether the system is integrable or not. The distributions indicate that we are still near integrability. However, the peak at $s = 0$ in $P(s)$ becomes smaller for $q = 0.9$, and for $s > 0$ the distribution already looks more like a Wigner distribution. Finally, for $s = 0.8$ universal level repulsion is fully developed, as can be seen in the graph of $N(s)$.

The results for the second system show the onset of level repulsion more beautifully in figure 2. For $q = 1.0$ we see here a Poisson distribution. As already mentioned, such absence of correlations between the levels of a one-degree-of-freedom system is an artifact of the sine function in the Hamiltonian. However, now the promised beauty of the transition to level repulsion can be appreciated: for $q = 0.99$ and $q = 0.98$ we see a crossover from Poisson to Wigner behaviour. For $q = 0.6$, $P(s)$ agrees very nicely with the Wigner distribution.

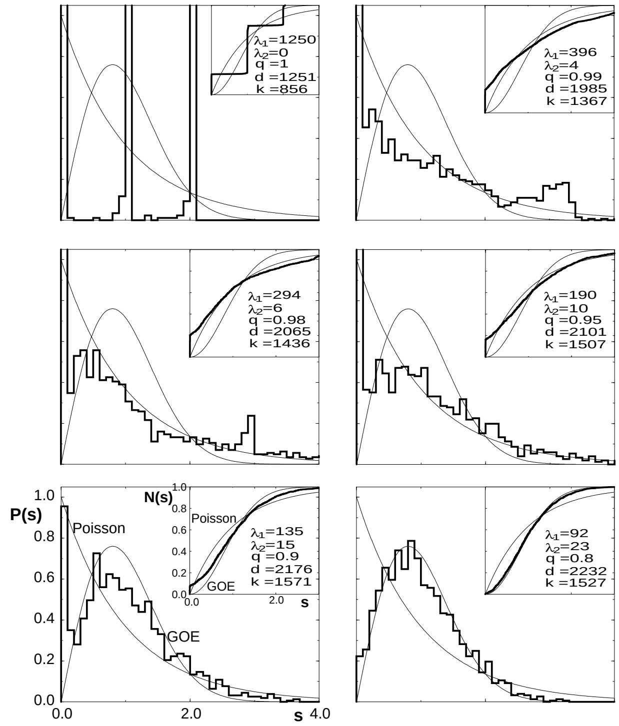

Figure 1. Level-spacing distribution $P(s)$ and its integral $N(s)$ for the Hamiltonian H_1 in different irreducible representations. The irreducible representations are labelled by λ_1 and λ_2 and by the parameter q, the dimension of the diagonalized matrix is d, and k energy eigenvalues in the range $-2.0 \leqslant E \leqslant 0.5$ have been taken into account for the statistics.

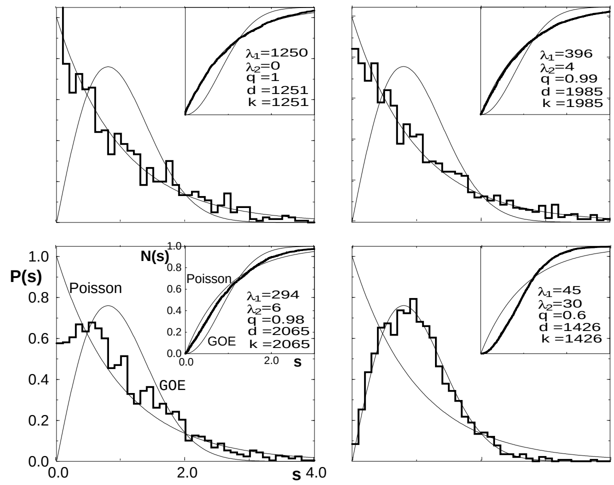

Figure 2. Level-spacing distribution $P(s)$ and its integral $N(s)$ for the Hamiltonian H_2 in different irreducible representations. The labels are as in figure 1.

4.3. Chaos and integrability in the classical limit

The above results indicate that in the classical limits corresponding to the irreducible representations used for the level statistics we should see a crossover from integrable to chaotic dynamics. This crossover is accompanied by the appearance of a new degree of freedom. This new degree of freedom is, however, very restricted for q near one (or near zero) such that chaos is not yet fully developed there.

We start by writing down the Hamilton functions H_1^{cl} and H_2^{cl} in canonical variables. The difference of the classical limits for $0 \leqslant q \leqslant 1$ then becomes obvious. In (4.4) and (4.5) there is a hidden restriction: in the classical limit two Casimir functions have been fixed. In canonical variables, i.e. in variables fulfilling canonical Poisson bracket relations

$$\{I_i, \phi_j\} = \delta_{ij} \qquad \text{for} \quad i, j = 1, 2, 3 \tag{4.8}$$

the two Hamilton functions are

$$\begin{aligned} H_1^{cl}(I_j, \phi_j) = {} & I_2 + 6(I_3^2 - I_2^2)\cos(2\phi_2) + 30\sqrt{A_q^2 - B_q^2}\cos(\phi_2)60A_qB_q(I_3\sin(\phi_2)\sin(\phi_3) \\ & -I_2\cos(\phi_2)\cos(\phi_3)) \end{aligned} \tag{4.9}$$

$$\begin{aligned} H_2^{cl}(I_j, \phi_j) = {} & \sin(100I_2) + (I_3^2 - I_2^2)\cos(2\phi_2) \\ & +200\sqrt{A_q^2 - B_q^2}\cos\phi_2 400A_qB_q(I_3\sin(\phi_2)\sin(\phi_3) - I_2\cos(\phi_2)\cos(\phi_3)). \end{aligned} \tag{4.10}$$

The coefficients A_q, B_q are given above in (3.34). Neither Hamiltonian depends on ϕ_1 and hence the classical system has the constant of motion $I_1 = y$, the classical analogue of the quantum constant Y. We may therefore work in a phase space spanned by $I_2 = t_3$, $I_3 = t$, ϕ_2, ϕ_3. With this (rather brutal form of a) symplectic reduction we are left with a dynamics with two degrees of freedom and $I_1 = y$ is a parameter with range $-\frac{q+1}{3} \leqslant y \leqslant \frac{2-q}{3}$. We have chosen $I_1 = y = \frac{2q-1}{3}$, which corresponds to the choice we made for the eigenvalue of the hypercharge Y in the quantum case. The coefficients A_q and B_q simplify to

$$\begin{aligned} A_q &= \frac{1}{2I_3}\sqrt{\left(q - \frac{1}{2} + I_3\right)\left(\frac{1}{2} - I_3\right)\left(\frac{1}{2} + I_3\right)} \\ B_q &= \frac{1}{2I_3}\sqrt{\left(-q + \frac{1}{2} + I_3\right)\left(\frac{1}{2} - I_3\right)\left(\frac{1}{2} + I_3\right)}. \end{aligned} \tag{4.11}$$

The compactness of the phase space is expressed by a limited range of the action variables I_j. When $I_1 = y = \frac{2q-1}{3}$, the range of $I_3 = t$ is

$$\begin{aligned} & q - \tfrac{1}{2} \leqslant t \leqslant \tfrac{1}{2} \qquad && \text{for} \quad \tfrac{1}{2} \leqslant q \leqslant 1 \\ & \tfrac{1}{2} - q \leqslant t \leqslant \tfrac{1}{2} \qquad && \text{for} \quad 0 \leqslant q \leqslant \tfrac{1}{2} \end{aligned} \tag{4.12}$$

and the range of $I_2 = t_3$ (when $I_3 = t$ is fixed)

$$-t \leqslant t_3 \leqslant t. \tag{4.13}$$

For initial conditions for the action variables which are inside the allowed range one can show that the trajectories always remain inside this range.

Now for $q = 1$ or $q = 0$ the allowed range for $I_3 = t$ shrinks to one point, $I_3 = t = \frac{1}{2}$. In these cases we remain with a single degree of freedom and its canonical pair of variables I_2 and ϕ_2. The coefficients A_q and B_q both vanish and the Hamilton functions become

$$H_{1,q=0}^{cl}(I_2, \phi_2) = H_{1,q=1}^{cl}(I_2, \phi_2) = I_2 + 6(\tfrac{1}{4} - I_2^2)\cos(2\phi_2) \tag{4.14}$$

$$H_{2,q=0}^{cl}(I_2, \phi_2) = H_{2,q=1}^{cl}(I_2, \phi_2) = \sin(100I_2) + (\tfrac{1}{4} - I_2^2)\cos(2\phi_2). \tag{4.15}$$

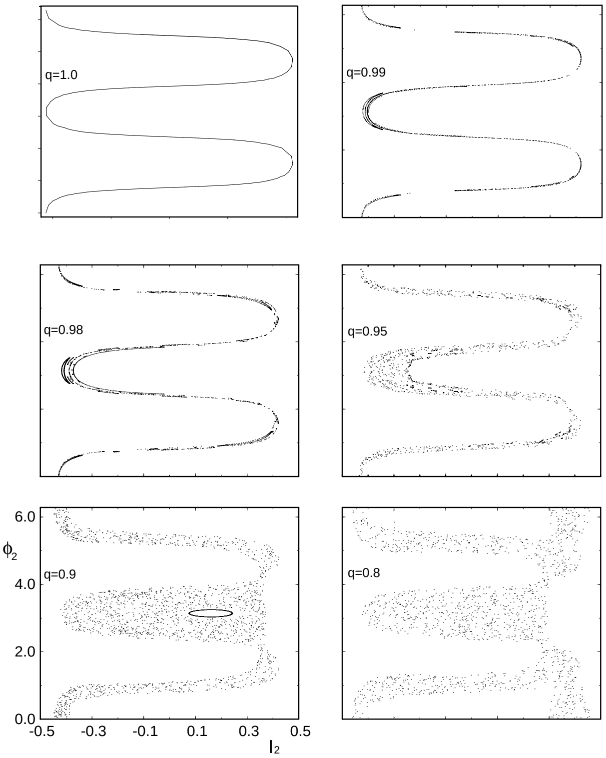

Figure 3. The upper-left corner panel shows the energy shell $H_1^{cl} = 0$ in the reduced two-dimensional phase space for $q = 0$. The other panels show phase portraits in the Poincaré surface of section $\phi_3 = 0$ in the energy shell $H_1^{cl} = 0$, for various values of q.

Having only one degree of freedom these Hamiltonian systems are, of course, integrable. However, if we vary the parameter q the range of $I_3 = t$ changes from a single point to a finite interval and the Hamiltonian system has two degrees of freedom. To demonstrate the appearance of chaos we have chosen to show numerically obtained phase portraits in Poincaré surfaces of section. Our systems have two degrees of freedom (if $q \neq 0, 1$), so an energy shell and a Poincaré surface of section generically have, respectively, dimension three and two. We shall restrict all following discussions to the energy shell $H_i(I_j \cdot \phi_k) = E = 0$ which lies at the centre of the energy range (the energy range is bounded from above and below). As a Poincaré surface we choose $\phi_3 = 0$ and project it onto the plane spanned by I_2 and ϕ_2 (no effort has been made to distinguish in which direction a trajectory pierces the Poincaré surface). Figure 3 illustrates the crossover from a one-freedom system to a chaotic two-freedom system for H_1^{cl}. The first panel, for $q = 1$, does not portray a Poincaré surface of section but shows the whole energy shell $E = 0$ in (reduced) phase space. The following panels show phase portraits in Poincaré surfaces of section as described above. For values of q very near to 1, the phase portraits are still confined to a narrow region around the energy shell for $q = 1$. It is only when q is considerably reduced that the phase portrait occupies a larger region in the (I_2, ϕ_2) plane and chaos occurs (very few periodic orbits can be seen). In figure 4 we have magnified regions from two of the phase portraits. In the upper case, for $q = 0.98$, the phase portrait is still confined to a narrow region and we see lots of stable tori. This fits well with the Poissonian level-spacing distributions we find in irreducible representations with the same value of q (see

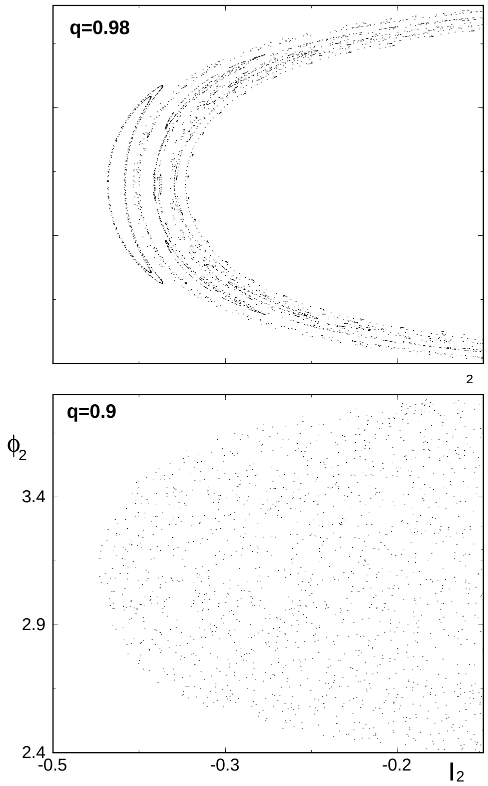

Figure 4. In these two panels we have magnified a region from two of the phase portraits from figure 3.

figure 1). In the lower case, with $q = 0.9$, the phase portrait has broadened and no stable tori remain visible. Again this fits the corresponding level-spacing distribution which is Wigner for the same value of q.

Figure 5 shows the energy shell $H_2^{cl} = E = 0$ for $q = 1$ and phase portraits in Poincaré surfaces of section for various values $q < 1$ for H_2^{cl}. Again, the energy shell for $q = 1$ is one-dimensional and all trajectories with energy $E = 0$ are confined to it. As q is reduced we have the same qualitative behaviour as for H_1^{cl}. The phase portraits broaden indicating that the three-dimensional volume of the energy shell grows. Again, chaos appears as soon as sufficient broadening has emerged, once again in correspondence with the results for the level statistics (see figure 2).

5. Conclusion

By considering two model systems with SU_3 generators as dynamical variables we have shown that statistical properties of quantum spectra depend not only on the form of the Hamiltonian but also on the representation of the SU_3 group relevant to the problem. One and the same Hamiltonian can exhibit features characteristic of integrability (clustering of the energy levels) as well as of chaotic systems (level repulsion). We showed that this can be attributed to different integrability properties of the classical limit, which in the case of SU_3 can be attained in many ways depending on the chosen sequences of irreducible representations.

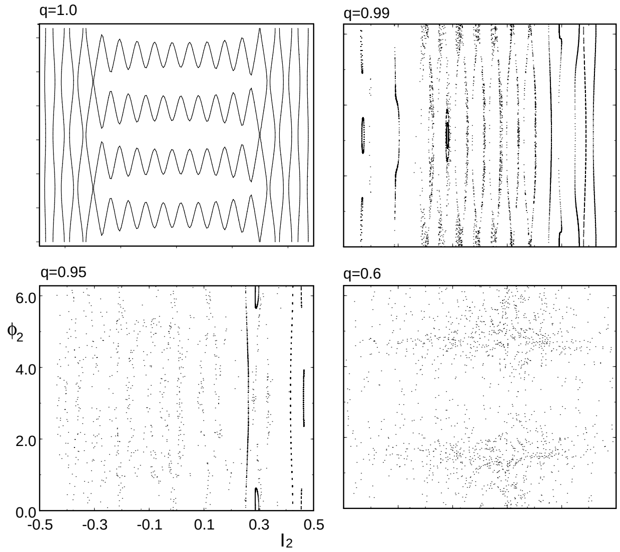

Figure 5. The upper-left corner panel shows the energy shell $H_2^{cl} = 0$ in the reduced two-dimensional phase space for $q = 0$. The other panels show phase portraits in the Poincaré surface of section $\phi_3 = 0$ in the energy shell $H_2^{cl} = 0$, for various values of q.

In particular, the dimensionality of the classical phase space depends on the way the classical limit is approached, and could be reduced from six (the generic case) to four (corresponding to degenerate irreducible representations). If there is one constant of motion the classical dynamics is integrable in the four-dimensional phase spaces ($q = 0$ and $q = 1$) but generically non-integrable in the six-dimensional phase spaces ($0 < q < 1$). Correspondingly, the spectra of the quantum Hamiltonian with one constant of motion only shows level repulsion for the irreducible representations $[\lambda_1; \lambda_2]$, with both $\lambda_1 \neq 0$ and $\lambda_2 \neq 0$.

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